Mean-variance portfolio selection with random parameters

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Market models with random parameters

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt \\ dS_i(t) = \mu(t)S_i(t)dt + S_i(t)\sigma(t)dW(t) \end{cases}$$

- Incomplete market
  - $(m+1)$ assets: $S_0(t), S_1(t), \cdots, S_m(t)$
  - $(m+d)$ sources of uncertainty $(d \geq 0)$

$$\overline{W}(t) \triangleq (W(t), B(t))$$

- Random parameter models
  - volatility smile; leptokurtic distribution.
  - more accurate but difficult to use.

Random parameters:

- Stochastic parameter models:

$$r(t,\omega) = r(t, y(t)), \quad \mu(t,\omega) = \mu(t, y(t)), \quad \sigma(t,\omega) = \sigma(t, y(t))$$

$$\begin{cases} dy(t) = \alpha(t, y(t))dt + \beta(t, y(t))d\overline{W}(t) \\ y(0) = y \end{cases}$$

e.g. Hull & White (1987), Stein & Stein (1991), Heston (1993)

- Parameter estimation:

$\mu(t), \sigma(t)$ estimated online from observations.

(Joint work with Xin Guo).

Example: Partially observed liabilities/contingent claims

Market model:

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt \\ dS_i(t) = \mu_i(t)S_i(t)dt + S_i(t)\sigma_i(t)dW(t) \end{cases}$$

$W(t)$ an $n-$dim S.B.M. $(m \leq n)$ with $F_t = \sigma\{W(s), 0 \leq s \leq t\}$.

Liability/Contingent claim

(F_t-measurable) liability/contingent claim ξ with payoff $\xi f(V(T))$

$$\text{e.g. } \xi = g(S(T))$$

where:

$$\begin{cases} V(t) = e^{Y(t)} \\ Y(t) = v + \gamma N(t) \end{cases}$$

Observations:

$$H(t) = \int_0^t C(s)Y(s)ds + \int_0^t \eta(s)dM(s)$$

Assume:

$v \sim N(v_0, \Sigma_0)$;

$v,\ W(t),\ M(t),\ N(t)$ mutually independent

η(t) invertible

Kalman Filtering

Given observations $H(s),\ 0 \le s \le t$: $Y(t) \sim N(\hat{Y}(t),\ \Sigma(t))$

$$\begin{cases} d\hat{Y}(t) = \Sigma(t)C(t)'(\eta(t)\eta(t)')^{-1}\eta(t)dQ(t) \\ \hat{Y}(0) = v_0 \end{cases}$$

$$\begin{cases} \dot{\Sigma}(t) = \gamma(t)\gamma(t)' - \Sigma(t)C(t)'(\eta(t)\eta(t)')^{-1}C(t)\Sigma(t) \\ \Sigma(0) = \Sigma_0 \end{cases}$$

$Q(t)$ (available from observations) given by:

$$Q(t) := \int_0^t \eta(s)^{-1}[dH(s) - C(s)\hat{Y}(s)ds]$$

$Q(t)$ and $W(t)$ are mutually indep. S.B.M. which generate

$$G_t := \sigma\{(W(s), H(s)), 0 \le s \le t\} = \sigma\{(W(s), Q(s)), 0 \le s \le t\}.$$

Optimal hedging of partially observed liabilities

$$\min_{\pi} E[(x(T)-\xi f(V(T)))^2]$$

$$\equiv \min_{\pi} E\left[(x(T)-\frac{\xi}{\sqrt{2\pi\Sigma(T)}}\int_{-\infty}^{\infty} f(e^x)e^{\frac{(x-\hat{Y}(T))^2}{2\Sigma(T)}}dx)^2 \right]$$

$$\begin{cases} dx(t)=[r(t)x(t)+(\mu(t)-r(t)1)'\pi(t)]dt+\pi(t)'\sigma(t)dW(t) \\ x(0)=x_0 \end{cases}$$

$\pi(\cdot)$ adapted to G_t.

Equivalent to problem with `complete information’

Reference:

A.E.B. Lim. Partially observed contingent claims. Working paper, IEOR Dept, U.C. Berkeley, 2003.

Basic issues

- How should we do portfolio selection when the parameters in the market model are stochastic processes?

- We examine this issue by studying the problem of mean-variance portfolio selection in continuous time using the framework of stochastic control and backward stochastic differential equations.

References

1. A.E.B. Lim & X.Y. Zhou. Mean-variance portfolio selection with random parameters in a complete market.

   Mathematics of Operations Research, Vol. 27 No. 1, pp 101-120, 2002.
2. A.E.B. Lim. Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market, 2002.

   (Accepted for publication in *Mathematics of Operations Research).*
3. A.E.B. Lim. Partially observed contingent claims.

   Working paper, IEOR Dept, U.C. Berkeley, 2003.

Basic setup

Complete probability space (Ω, F, P)

Standard $(m+d)-$dimensional Brownian motion on $[0, T]$:
$\overline{W}(t) := (W(t), B(t))$.

$\{F_t\}_{t\geq 0}$ the filtration generated by $\overline{W}(t),$ augmented by the P-null sets of F.

Outline

1. Motivations.
2. Backward stochastic differential equations.
    - Nonlinear BSDEs.
    - Conditions for solvability.
3. Mean-variance problem.
4. Main results.
5. Conclusion.

Nonlinear BSDEs: General results

- Pardoux & Peng (1990), Duffie & Epstein (1992).

$$\begin{cases} dx(t) = f(t, x(t), z(t))dt + z(t)d\overline{W}(t) \\ x(T) = \xi \end{cases}$$

ξ an F_T-measurable, square integrable r.v.

- A solution is a pair of square integrable, adapted processes *(x(t), z(t))* which satisfy the SDE.
- Solvability under similar conditions as (forward) SDEs:
  - global Lipschitz continuity.
  - linear growth conditions.
  - …
- Proof based on a fix point theorem.

Example: Contingent claim replication

Random liability ξ at time T.

(F_T – measurable, square integrable r.v.).

e.g. Lookback option: $\xi = \max_{0 \le t \le T} S(t)$

Stock and bond:

$$\begin{cases} \mathrm{d}S_1(t) = S_1(t)[\mu(t)dt + \sigma(t)dW(t)] \\ dS_0(t) = r(t)S_0(t)dt \end{cases}$$

Is there a way to invest in the stock market so that the total earnings (from trading) will cover the liability?

i.e. find an investment rule $\pi(\cdot)$ such that the resulting wealth process $x(\cdot)$ satisfies $x(T) = \xi$.

Written in mathematics …

Find a pair of processes $(x(\cdot), \pi(\cdot))$ which are adapted, square integrable and satisty

$$\begin{cases} dx(t) = [r(t)x(t) + (\mu(t) - r(t)1)'\pi(t)dt + \pi(t)'\sigma(t)dW(t) \\ x(T) = \xi \end{cases}$$

Can this be done?

Girsanov transformation + Martingale Rep. Theorem

$\Rightarrow$ existence and uniqueness.

If ξ and/or $r(\cdot)$ are random, generally need $\pi \neq 0$.

Outline

1. Motivations.
2. Backward stochastic differential equations.
3. Mean-variance problem.
   - formulation
   - solvability of the stochastic Riccati equation (complete & incomplete markets).
4. Main results.
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Mean-variance portfolio selection: Problem statement

For a given initial wealth x_0, find an investment portfolio $\pi(\cdot)$ that results in an expected return of c with minimum variance. Markowitz (1952).

$$\begin{cases} \min\limits_{\pi(\cdot)\in L_F^2} Var\ x(T) \equiv E(x(T)-c)^2 \\ \text{subject to:} \\ dx(t) = [r(t)x(t) + (\mu(t) - r(t)1)'\pi(t)dt + \pi(t)'\sigma(t)dW(t) \\ x(0) = x_0 \\ Ex(T) = c \end{cases}$$

The key to solving the m-v- problem is the following dynamic optimization problem.

$$\begin{cases} \min_\pi E[(x(T)-c)^2 + \lambda(x(T)-c)] \\ dx(t) = [r(t)x(t) + (\mu(t) - r(t)1)'\pi(t)dt + \pi(t)'\sigma(t)dW(t) \\ x(0) = x_0 \end{cases}$$

The Hamilton-Jacobi-Bellman PDE can not be used when $r(\cdot), \mu(\cdot), \sigma(\cdot)$ are random processes.

Sufficient condition for solvability and expression for optimal policy $\pi(\cdot)$ obtained by using:

`completion of squares' + BSDEs

Stochastic Riccati Equation

- Sufficient condition for solvability of m-v problem is existence and uniqueness of solutions of SRE:

$$\begin{cases} dp(t) = -[A(t)p(t) + B(t)\Lambda(t) - \dfrac{1}{p(t)}\Lambda_1(t)'\Lambda_1(t)]dt + \Lambda(t)'d\overline{W}(t) \\ p(T) = 1 \\ p(t) > 0, \forall t \in [0,T) \end{cases}$$

Complete solution of the mvp is given in terms of solution $(p(\cdot), \Lambda(\cdot))$ of SRE.

- Nonlinear, singular BSDE which is neither globally Lipschitz continuous nor linearly growing.

  i.e. usual existence and uniqueness results do not apply.

Is the SRE solvable? (Complete market case)

Theorem 1: When the market is complete, the SRE has a unique solution.

Lim & Zhou (2001)

Key observation in proof: 1/p(t) is the `solution' of a linear BSDE.

1. Linear BSDE:

$$\begin{cases} dy(t) = -[-A(t)y(t) + B(t)z(t)]dt + z(t)'dW(t) \\ y(T) = 1 \end{cases}$$

Existence & uniqueness of $(y(\cdot), z(\cdot))$ follows from standard theory.

In addition $y(t) \geq \delta > 0, \forall t \in [0, T]$.

2. Stochastic Riccati equation: Define

$$(p(t),\Lambda(t)) := \left(\frac{1}{y(t)}, -\frac{z(t)}{y(t)^2}\right)$$

Ito's formula

$$\Rightarrow \begin{cases} dp(t) = -[A(t)p(t) + B(t)\Lambda(t) - \dfrac{1}{p(t)}\Lambda(t)'\Lambda(t)]dt + \Lambda(t)'dW(t) \\ p(T) = 1 \end{cases}$$

which is the original SRE.

Is the SRE solvable? (Incomplete market case)

Theorem 2: Under ‘standard assumptions’, the SRE associated with the m-v problem has a unique solution.

Lim (2002).

M. Kobylanski, Annals of Probability (2000):

$$\begin{cases} dp(t) = -[A(t)p(t) + B(t)\Lambda(t) - \dfrac{1}{R(t)}\Lambda(t)'Q(t)\Lambda(t)]dt + \Lambda(t)'d\overline{W}(t) \\ p(T) = 1 \end{cases}$$

SRE corresponds to the case $R(t) = p(t)$.

Iteration:

$$R(t) = p_0(t) \rightarrow (p_1, \Lambda_1) \rightarrow (p_2, \Lambda_2) \rightarrow \cdots$$

Solution of SRE:

$$(p, \Lambda) = \lim_{k \to \infty} (p_k, \Lambda_k)$$

(Under certain sufficient conditions on p_0)

Outline

1. Motivations.
2. Backward stochastic differential equations.
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4. Main results.
   - optimal policy & efficient frontier.
   - numerical solutions.
5. Conclusion.

Optimal policy & efficient frontier: Complete market

Risk-free security: $(g(\cdot), \varsigma(\cdot))$ the solution of:

$$\begin{cases} dg(t) = \{r(t)g(t) + [\mu(t) - r(t)1]'(\sigma(t)^{-1})'\varsigma(t)\}dt + \varsigma(t)'dW(t) \\ g(T) = 1 \end{cases}$$

$g(\cdot) \equiv$ security that pays \$1 at time T.

$(\sigma(t)^{-1})'\varsigma(t) \equiv$ associated investment policy.

For any c, define:

$$k = \frac{c - p(0)g(0)x_0}{1 - p(0)g(0)^2} > 0$$

Optimal policy:

$$\pi^*(t) = (\sigma(t)^{-1})'[\theta(t) + \frac{\Lambda(t)}{p(t)}](kg(t) - x(t)) + k(\sigma(t)^{-1})'\varsigma(t)$$

Efficient frontier:

$$\sigma_{\bar{x}(T)} = \sqrt{\frac{p(0)g(0)^2}{1 - p(0)g(0)^2}}[E\bar{x}(T) - \frac{x_0}{g(0)}]$$

Optimal policy & efficient frontier: Incomplete market

Risk-free security in fictitiously completed market:

$$\begin{cases} dg(t) = [r(t)g(t) + \bar{\theta}(t)'\varsigma(t)]dt + \varsigma(t)'d\bar{W}(t) \\ g(T) = 1 \end{cases}$$

$$\bar{\theta}(t)' = \left[(\mu(t) - r(t)1)'(\sigma(t)^{-1}), \frac{\Lambda_2(t)'}{p(t)} \right]$$

$$\begin{bmatrix} \sigma(t)^{-1}\varsigma_1(t) \\ \dfrac{1}{p(t)}\varsigma_2(t) \end{bmatrix} \equiv$$

Hedging portfolio for riskless security in the fictitious market that pays $1.

For any given c:

$$k \triangleq \frac{c - p(0)g(0)x_0}{1 - M - p(0)g(0)^2}$$

$$M \triangleq E\int_0^T p\varsigma[I - \sigma'(\sigma\sigma')^{-1}\sigma]\varsigma\,dt$$

Optimal policy:

$$\pi^*(t) = (\sigma(t)^{-1})'[\theta(t) + \frac{\Lambda_1(t)}{p(t)}](kg(t) - x(t)) + k(\sigma(t)^{-1})'\varsigma_1(t)$$

Efficient frontier:

$$(\sigma_{\bar{x}(T)})^2 = \alpha\left[E\,\bar{x}(T) - \frac{p(0)g(0)}{M + p(0)g(0)^2}x_0\right]^2 + \beta$$

<u>Example</u>: Stochastic volatility models of Hull & White (1987), Stein & Stein (1991), Heston (1993).

$$\begin{cases} dS_1(t) = S_1(t)\left[\mu(t)dt + \sigma(t)dW(t)\right] \\ dS_0(t) = r(t)S_0(t)dt \end{cases}$$

$$r(t,\omega) \triangleq r(t,y(t)),\ \ \mu(t,\omega) \triangleq \mu(t,y(t)),\ \ \sigma(t,\omega) \triangleq \sigma(t,y(t))$$

$$\theta(t,\omega) \triangleq \frac{\mu(t,y(t)) - r(t,y(t))}{\sigma(t,y(t))}$$

$$\begin{cases} dy(t) = \alpha(t,y(t))dt + \beta(t,y(t))dB(t) \\ y(0) = \mathrm{y} \end{cases}$$

Linear parabolic PDE:

$$\begin{cases} Z_t + (2r - |\theta|^2)(t,y)Z + \alpha(t,y)Z_y + \frac{1}{2}\beta(t,y)^2 Z_{yy} = 0 \\ \qquad y \in R,\ 0 \le t < T \\ Z(T,y) = 1. \end{cases}$$

Solution of SRE:

$$\begin{cases} p(t,\omega) = Z(t,y(t)) \\ \begin{bmatrix} \Lambda_1(t,\omega) \\ \Lambda_2(t,\omega) \end{bmatrix} = \begin{bmatrix} 0 \\ Z_y(t,y(t))\beta(t,y(t)) \end{bmatrix} \end{cases}$$

Conclusions

- Market models with random parameters.
- Solvability of the stochastic Riccati equation (complete & incomplete markets).
- Optimal portfolio and efficient frontier.
- Related research:

  \- constrained portfolios.

  \- partially observed liabilities.

  \- online parameter estimation and portfolio choice (with X. Guo)

  \- nonlinear utility/loss functions.