Stochastic Models of Energy Commodity Prices and Their Applications: Mean-reversion with Jumps and Spikes

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Abstract

In this paper, we present several mean-reversion jump diffusion models to describe energy commodity spot prices. We incorporate multiple jumps, regime-switching and stochastic volatility in these models. Prices of various energy commodity derivatives are obtained under each model. We show how the electricity derivatives can be used to evaluate generation and transmission capacity. We also show for our price models, how to determine the value of investment opportunities and the threshold value above which a firm should invest.

1 Introduction

Energy commodity markets grow rapidly as electricity market reform are spreading in the United States and around the world. The volume of trade for electricity reported by US power marketers has surged from 27 million MWh in 1995 to 1,195 million MWh in 1997 (Source: Edison Electric Institute). This trend inevitably exposes the portfolios of generating assets

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and various supply contracts held by traditional electric utility companies to market price risk. It has so far changed and will continue to change not only the way a utility company operates and manages its physical assets such as power plants, but also the way it values and selects potential investment projects. In general, risk management and asset valuation needs require understanding and sophisticated modeling of commodity spot prices. Among all the energy commodities, modeling price behavior of electricity poses the biggest challenge for researchers and practitioners. In the summer of 1998, prices of electricity fluctuated from $0/MWh to $7000/MWh. It is not uncommon to see a 150% implied volatility in traded electricity options. Schwartz (1996) and Miltersen and Schwartz (1998) investigated several stochastic models for commodity spot prices and performed an empirical analysis based on copper, gold and crude oil price data. They found that stochastic convenience yields could explain the term structure of forward prices and demonstrated the implications to hedging and real asset valuation by different models. In our paper, we examine a broader class of stochastic models which can be used to model behavior in commodity prices such as jump and stochastic volatility, in addition to stochastic convenience yield. We feel that models with jumps and stochastic volatility are particularly suitable for modeling of the spot price processes of nearly non-storable commodities such as electricity.

While some energy commodities, such as crude oil, may be properly modeled as traded securities, the nonstorability of electricity makes such an approach inappropriate. However, we can always view the spot price of a commodity as a state variable. All the physical contracts/financial derivatives on this commodity are therefore contingent claims on the state variable. We model the spot price processes of energy commodities as affine jump-diffusion processes, introduced in Duffie and Kan (1996). Affine jump-diffusion processes are flexible enough to allow us to capture the special characteristics of commodity prices such as mean-reversion, seasonality, and "spikes" (i.e. upward jumps followed shortly by downward jumps). More importantly, we are able to compute the prices of various energy commodity derivatives for the assumed underlying affine jump diffusion price processes by applying transform analysis (see Duffie, Pan and Singleton (1998)). In this paper, we consider not only the usual affine jump-diffusion models but also a regime-switching mean-reversion jump-diffusion model. That model is used to capture the random switches between "abnormal" and "normal" equilibrium states of supply and demand for a commodity.

Jumps and spikes in the price of a commodity also have significant impacts on the values of investment opportunities and the optimal timing of investment in a production facility for the commodity. As we will see in section four, a downward jump in the investment value process reduces the value of the investment opportunity and shorten the expected waiting time to invest. A regime-switching type of value process, on the other hand, would restore some of the value of investment opportunity and make it advantageous to wait longer. Moreover, when the investment value process is regime-switching, investment in a project should never occur in the "low" state regardless the investment value; while in the "high" state, investment is indicated when the investment value exceeds a certain threshold value.

The remainder of the paper is organized as follows. In the next section, we propose three alternative models for energy commodity spot-price processes and compute the transform functions needed for contingent-claims pricing. In section three, we present some illustrative examples of the models specified in section two. We obtain the pricing formulae of several energy commodity derivatives in the examples. The comparisons of the prices of energy commodity derivatives under different models are also shown. We provide a heuristic method for estimating the process parameters by matching moment conditions of the historical spot prices and calibrating the parameters to traded options prices. In section four, we choose electricity as an concrete example to illustrate how different types of generation assets can be valued as electricity options. We also demonstrate the implication of jumps and spikes in the investment value process to the value of investment opportunity and the timing of investment decision. We conclude by making some observations and pointing out future research directions.

2 Mean-Reverting Jump-Diffusion Models

The most noticeable price behavior of energy commodities is the mean-reverting effect. When the price of a commodity is high, its supply tends to increase thus putting a downwards pressure on the price; when the spot price is low, the supply of the commodity tends to decrease thus providing an upwards lift to the price. Another salient effect in energy commodity prices is the presence of price jumps and spikes. This is particularly prominent in the case where massive storage of a commodity is not economically viable and demand exhibits low elasticity. A perfect example of such price behavior would be electricity which is almost non-storable. Figure (1) plots the historical on-peak electricity spot prices in Texas (ERCOT) and at the California and Oregon border (COB). The fundamental economic principle says that in a competitive market the spot price of a commodity is determined by the crossing

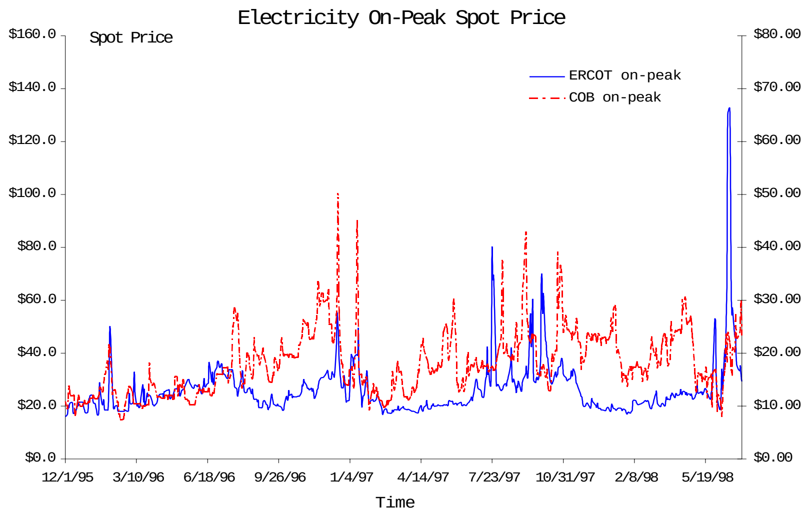

Figure 1: Electricity Historical Spot Prices

point of its aggregate supply and demand curves as illustrated in Figure (2). The reason we see such jumpy behaviors in electricity spot prices is due to the fact that a typical aggregate supply curve for electricity in a region exhibits kinks at certain capacity level and the curve has a steep upward slope beyond that capacity level. A snap shot of the marginal cost curve of the supply resource stack for electricity in western US is shown in Figure (3). A forced outage of a major power plant or sudden surging demand will either shift the supply curve to the left or lift up the demand curve therefore causing a price jump. When the contingency making the spot price to jump high is of short-term nature, the high price will quickly fall back down to the normal range as the contingency disappears therefore causing a spike in the commodity spot price process. In the summer of 1998, we observed the spot price of electricity in the Eastern and Midwestern US jumped from \$50/MWh to \$7000/MWh because of the unexpected unavailability of some plants and congestion in key transmission lines. Within a couple of days the price fell back to the \$50/MWh range. Although the mean reversion is well studied, there is little work examining the implication of the combined effects of jumps and spikes to risk management and asset valuation.

In this paper, we will examine the following three types of mean-reverting jump-diffusion models for modeling energy commodity spot price process.

1. Mean-reverting jump-diffusion process with deterministic volatility.

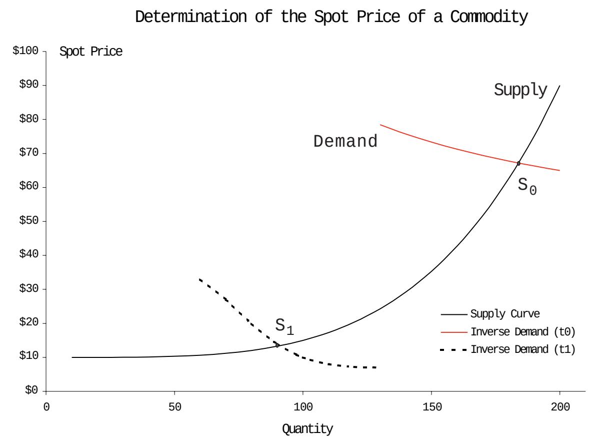

Figure 2: Determination of Spot Prices

2. Mean-reverting jump-diffusion process with regime-switching.

3. Mean-reverting jump-diffusion process with stochastic volatility.

We consider two types of jumps in all the above models. While our analytical approach could handle multiple types of jumps, with properly chosen intensities two types of jumps suffice to mimic the jumps and/or spikes in the price process of energy commodities. The case of only one type of jump is included as a special case when the intensity of type-2 jump is set to zero.

In addition to the commodity price process under consideration, we also jointly specify another factor process which can be correlated with the commodity price. This additional factor process could be the price process of another commodity or it could be the underlying physical demand process of a utility company for an energy commodity. In the case of electricity, the additional factor can be used to specify the spot price of the generating fuel such as natural gas. A jointly specified spot price process of the generating fuel is essential for risk management involving cross commodity risks between electricity and the fuel. There is empirical evidence demonstrating a positive correlation between the price of electricity and the price of generating fuel in certain geographic region during certain times of a year. In all models the risk free interest rate, r, is assumed to be deterministic.

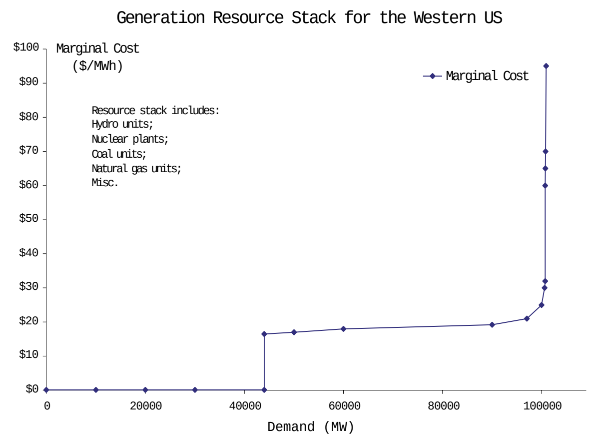

Figure 3: Generation Stack for Electricity in a Region

2.1 Model 1: A mean-reverting deterministic volatility process with two types of jumps

We start with specifying spot price of an energy commodity as a mean-reverting jump-diffusion process with two types of jumps. Let $X_t = \ln S_t^e$ where S_t^e is the energy commodity spot price, e.g. electricity spot price. Y_t is the factor process which, in the case of modeling electricity spot price, can be used to specify the spot price of a generating fuel, e.g. natural gas. In this formulation, we have type-1 jump representing the upwards jumps and type-2 jump representing the downwards jumps. By choosing the intensity functions properly for the jump processes, we can mimic the spikes in the price process of various energy commodities. Assume that, under regularity conditions, X_t and Y_t are strong solutions to the following stochastic differential equations (SDE) under the risk-neutral measure Q,

$$\begin{aligned} d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} \kappa_1(t)(\theta_1(t) - X_t) \\ \kappa_2(t)(\theta_2(t) - Y_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1(t) & 0 \\ \rho(t)\sigma_2(t) & \sqrt{1-\rho(t)^2}\sigma_2(t) \end{pmatrix} dW_t \\ &\quad + \sum_{i=1}^{2} \Delta Z_t^i \end{aligned} \tag{2.1}$$

where $\kappa_1(t)$ and $\kappa_2(t)$ are the mean-reverting coefficients; $\theta_1(t)$ and $\theta_2(t)$ are the long term means; $\sigma_1(t)$ and $\sigma_2(t)$ are instantaneous volatility rates of X and Y; W_t is an $\mathcal{F}_t$-adapted standard Brownian motion under Q in $\Re^2$; Z^j is a compound Poisson process in $\Re^2$ with the Poisson arrival intensity being $\lambda_j(t)$ $(j = 1, 2)$. ΔZ^j represents the random jump size in

$\Re^2$. Let $\phi_J^j(c_1, c_2, t) \equiv \int_{R^2} \exp(c \cdot \Delta Z_t^j) dv^j(z)$ denote the transform of the jump size of type-j ($j = 1, 2$) jump.

The transform function Define the generalized transform function as

$$\varphi(u, X_t, Y_t, t, T) \equiv E^Q[e^{-r(T-t)} \exp(u_1 X_T + u_2 Y_T) \mid \mathcal{F}_t]$$

for a fixed time T where $u \equiv (u_1, u_2) \in C^2$. The transform function φ is well-defined at a given u under regularity conditions on $\kappa_*(t)$, $\theta_*(t)$, $\sigma_*(t)$, $\lambda_*(t)$, and $\phi_J^*(c_1, c_2, t)$.

Under technical regularity conditions, $\varphi \cdot e^{-rt}$ is a martingale under the risk-neutral measure Q since it is a $\mathcal{F}_t-$conditional expectation of a single random variable $\exp(u_1 X_T + u_2 Y_T)$.[1] Therefore the drift term of $\varphi \cdot e^{-rt}$ is zero. Applying Ito's lemma for complex function, we observe that φ needs to satisfy the following fundamental partial differential equation (PDE)

$$\mathcal{D}f - rf = 0 \tag{2.2}$$

where

$$\mathcal{D}f \equiv \partial_t f + \mu_X \cdot \partial_X f + \frac{1}{2} tr(\partial_X^2 f \Sigma \Sigma^T) + \sum_{j=1}^{2} \lambda_j(t) \int_{R^2} [f(X_t + \Delta Z_t^j, t) - f(X_t, t)] dv_j(z)$$

It is conjectured that, under regularity conditions, φ takes the form:

$$\varphi(u, X_t, Y_t, t, T) = \exp[\alpha(t, u) + \beta_1(t, u) X_t + \beta_2(t, u) Y_t] \tag{2.3}$$

where α and $\beta \equiv [\beta_1 \ \beta_2]'$ satisfy the complex-valued ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\beta(t, u) + B(\beta(t, u), t) = 0, \qquad & \beta(0, u) = u \\ \frac{d}{dt}\alpha(t, u) + A(\beta(t, u), t) = 0, \qquad & \alpha(0, u) = 0 \end{aligned} \tag{2.4}$$

with $A(\cdot, \cdot) : C^2 \times R \to C^1$ and $B(\cdot, \cdot) : C^2 \times R \to C^2$ being

$$\begin{aligned} A(\beta, t) &= \sum_{i=1}^{2} [\kappa_i \theta_i \beta_i + \tfrac{1}{2} \sigma_i^2 \beta_i^2] + \rho \sigma_1 \sigma_2 \beta_1 \beta_2 - r + \sum_{j=1}^{2} \lambda_j(t) (\phi_J^j(\beta_1, \beta_2, t) - 1) \\ B(\beta, t) &= \begin{pmatrix} \kappa_1 \beta_1 \\ \kappa_2 \beta_2 \end{pmatrix} \end{aligned} \tag{2.5}$$

[1] For a formal proof see [7].

Now, we integrate α and β out with the corresponding initial conditions to get solutions (2.6). Then we have

$$\varphi(u, X_t, Y_t, t, T) = \exp(\alpha(t,u) + \beta_1(t,u)X_t + \beta_2(t,u)Y_t)$$

where

$$\begin{aligned}
\beta_1(t,[u_1,u_2]') &= u_1 \exp(-\textstyle\int_t^T \kappa_1(s)ds) \\
\beta_2(t,[u_1,u_2]') &= u_2 \exp(-\textstyle\int_t^T \kappa_2(s)ds) \\
\alpha(t,u) &= \int_t^T (\sum_{i=1}^{2}[\kappa_i(s)\theta_i(s)\beta_i(s,u) + \frac{1}{2}\sigma_i^2(s)\beta_i^2(s,u)] + \rho(s)\sigma_1(s)\sigma_2(s)\beta_1(s,u)\beta_2(s,u) \\
&\quad -r + \sum_{j=1}^{2} \lambda_j(s)(\phi_J^j(\beta_1(s,u),\beta_2(s,u),s) - 1))ds
\end{aligned} \tag{2.6}$$

2.2 Model 2: A regime-switching mean-reverting process with two types of jumps

To motivate this model, we consider again the case of electricity in which the forced outages of generation plants or unexpected contingencies in transmission networks often result in abnormally high spot prices for a short time period and then a quick price fall-back. In order to capture the phenomena of spot prices switching between "high" and "normal" states, we extend model 1 to a Markov regime-switching model which we will describe in detail below.

Let U_t be a continuous-time two-state Markov chain

$$dU_t = 1_{U_t=0} \cdot \delta(U_t)dN_t^{(0)} + 1_{U_t=1} \cdot \delta(U_t)dN_t^{(1)} \tag{2.7}$$

where $N_t^{(i)}$ is a Poisson process with arrival intensity $\lambda^{(i)}$ $(i = 0,1)$ and $\delta(0) = -\delta(1) = 1$. We next define the corresponding compensated continuous-time Markov chain $M(t)$ as

$$dM_t = -\lambda(U_t)\delta(U_t)dt + dU_t \tag{2.8}$$

The joint specification of electricity and the generating fuel price processes is

$$\begin{aligned}
d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} \kappa_1(t)(\theta_1(t) - X_t) \\ \kappa_2(t)(\theta_2(t) - Y_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1(t) & 0 \\ \rho(t)\sigma_2(t) & \sqrt{1-\rho(t)^2}\sigma_2(t) \end{pmatrix} dW_t \\
&\quad + \sum_{i=1}^{2} \Delta Z_t^i + \iota(U_{t-})dM_t
\end{aligned} \tag{2.9}$$

where $\{\iota(i) \equiv (\iota_1(i), \iota_2(i))'; i = 0, 1\}$ denotes the random jumps in state variables when regime-switching occurs.

The transform function Let $F^i(\overline{x}, t)$ $(i = 0, 1)$ denote

$$E[e^{-r(T-t)} \exp(u_1 X_T + u_2 Y_T) | X_t = x, Y_t = y, U_t = i]$$

where U_t is the Markov regime state variable. The infinitesimal generator $\mathcal{D}$ of F^i is given by

$$\begin{aligned} \mathcal{D}F^0(\overline{x}, t) &= dF^0(\overline{x}, t) + \lambda^{(0)} \int_{R^2} [F^1(\overline{x} + \iota(0), t) - F^0(\overline{x}, t)] dv_t(\iota(0)) \\ \mathcal{D}F^1(\overline{x}, t) &= dF^1(\overline{x}, t) + \lambda^{(1)} \int_{R^2} [F^0(\overline{x} + \iota(1), t) - F^1(\overline{x}, t)] dv_t(\iota(1)) \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} dF^i(\overline{x}, t) &= F^i_t + F^i_{\overline{x}} \cdot \mu^i(\overline{x}, t) + \frac{1}{2} tr[F^i_{\overline{x}\overline{x}} \sigma^i(\overline{x}, t) \sigma^i(\overline{x}, t)^T] \\ &\quad + \sum_{j=1}^{2} \lambda^i_j(\overline{x}, t) \int_{R^2} [F^i(\overline{x} + z, t) - F^i(\overline{x}, t)] dv_{j,t}(z) \\ (i &= 0, 1 \text{ is the regime state variable}) \end{aligned}$$

For the special case where the jump diffusion processes are the same in all regime states, the solution to (2.10) assumes the following form,

$$\begin{aligned} F^0(x, y, t) &= \exp(\alpha_0(t) + \beta_1(t)x + \beta_2(t)y) \\ F^1(x, y, t) &= \exp(\alpha_1(t) + \beta_1(t)x + \beta_2(t)y) \end{aligned} \tag{2.11}$$

Substituting (2.11) into $\mathcal{D}F^i(x, y, t) - rF^i(x, y, t) = 0$ and let $\alpha(t) \equiv \alpha(t, u) \equiv (\alpha_0(t, u), \alpha_1(t, u))'$, $\beta(t) \equiv \beta(t, u) \equiv (\beta_1(t, u), \beta_2(t, u))'$, we get the following ordinary differential equations (ODEs)

$$\begin{aligned} &\frac{d}{dt}\beta(t, u) + B(\beta(t, u), t) = 0, \qquad && \beta(0, u) = u \\ &\frac{d}{dt}\alpha(t, u) + A(\alpha(t, u), \beta(t, u), t) = 0, \qquad && \alpha(0, u) = 0 \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} A(\alpha(t), \beta(t), t) &= \begin{pmatrix} A_1(\beta(t), t) + \lambda^{(0)}[\exp(\alpha_1(t) - \alpha_0(t))\phi_{\iota(0)}(\beta(t), t) - 1] \\ A_1(\beta(t), t) + \lambda^{(1)}[\exp(\alpha_0(t) - \alpha_1(t))\phi_{\iota(1)}(\beta(t), t) - 1] \end{pmatrix} \\ B(\beta(t), t) &= \begin{pmatrix} \kappa_1(t)\beta_1(t) \\ \kappa_2(t)\beta_2(t) \end{pmatrix} \end{aligned} \tag{2.13}$$

with

$$A_1(\beta(t),t) = \sum_{i=1}^{2}[\kappa_i\theta_i\beta_i + \frac{1}{2}\sigma_i^2\beta_i^2] - \rho\sigma_1\sigma_2\beta_1\beta_2 - r + \sum_{j=1}^{2}\lambda_j(\phi_J^j(\beta,t) - 1)$$

$\phi_J^j(\beta,t)$ and $\phi_{\iota(\cdot)}(\beta(t),t)$ are transform functions of the random variables representing jump size in state variables within a regime and associated with the regime-switching, respectively. When ODEs (2.12) do not have closed-form solution, we need to numerically solve (2.12) and obtain the value of the transform function.

2.3 Model 3: A mean-reverting stochastic volatility process with two types of jumps

We consider a three factor affine process with two types of jumps in this model. Once again, we motivate the model in the setting of modeling the electricity spot price. Consider X_t and Y_t to be the logarithm of the spot price of electricity and generating fuel, e.g. natural gas, respectively. V_t can be thought as a factor which is proportional to the aggregate regional demand process. Weather conditions such as unusual heat waves usually cause simultaneous jumps in both electricity price and the aggregate load; while the plant or transmission line outages will likely cause jumps in electricity spot price only. There is also empirical evidence alluding to the fact that the volatility of electricity price is high when the aggregate demand surges and vice versa. With proper regularity conditions, there exists a Markov process which is the strong solution to the following stochastic differential equation (SDE) under the risk neutral measure Q.

$$d\begin{pmatrix} X_t \\ V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} \kappa_1(t)(\theta_1(t) - X_t) \\ \kappa_V(t)(\theta_V(t) - V_t) \\ \kappa_2(t)(\theta_2(t) - Y_t) \end{pmatrix} dt + \begin{pmatrix} \sqrt{V_t} & 0 & 0 \\ \rho_1(t)\sigma_2(t)\sqrt{V_t} & \sqrt{1-\rho_1^2(t)}\sigma_2(t)\sqrt{V_t} & 0 \\ \rho_2(t)\sigma_3(t)\sqrt{V_t} & 0 & \sigma_3(t) \end{pmatrix} dW_t + \sum_{i=1}^{2}\Delta Z_t^i \tag{2.14}$$

where W is an $\mathcal{F}_t$-adapted standard Brownian motion under Q in $\Re^3$; Z^i is a compound Poisson process in $\Re^3$ with the Poisson arrival intensity being $\lambda^i(X_t, V_t, Y_t, t)$ $(i = 1, 2)$. We model the spiky behavior by assuming that the intensity function of type-1 jump is only a function of time t, denoted by $\lambda^{(1)}(t)$ and the intensity of type-2 jump is a function of V_t, i.e. $\lambda^{(2)}(X_t, V_t, t) = \lambda_2(t)V_t$. Let $\phi_J^j(c_1, c_2, t) \equiv \int_{R^2} \exp(c \cdot \Delta Z_t^j) dv^j(z)$ denote the transform of the jump size of type-j $(j = 1, 2)$ jump.

The transform function Following similar arguments to those used in Model 1, we know that the transform function

$$\varphi(u, X_t, V_t, Y_t, t, T) \equiv E^Q[e^{-r(T-t)} \exp(u_1 X_T + u_2 V_T + u_3 Y_T) \mid \mathcal{F}_t]$$

is of form

$$\varphi(u, X_t, V_t, Y_t, t, T) = \exp(\alpha(t, u) + \beta_1(t, u) X_T + \beta_2(t, u) V_T + \beta_3(t, u) Y_T) \tag{2.15}$$

where $\alpha(u, t)$ and $\beta(u, t) \equiv (\beta_1(u, t), \beta_2(u, t), \beta_3(u, t))'$ are solutions to the following ordinary differential equations

$$\begin{aligned} \frac{d}{dt}\beta(t, u) + B(\beta(t, u), t) = 0, &\qquad \beta(0, u) = u \\ \frac{d}{dt}\alpha(t, u) + A(\beta(t, u), t) = 0, &\qquad \alpha(0, u) = 0 \end{aligned} \tag{2.16}$$

with $A(\cdot, \cdot) : C^3 \times R \to C^1$ and $B(\cdot, \cdot) : C^3 \times R \to C^3$ being

$$\begin{aligned} A(\beta, t) &= -r + \sum_{i=1}^{3} \kappa_i \theta_i \beta_i + \tfrac{1}{2}\beta_3^2 \sigma_3^2(t) + \lambda_1(t)(\phi_J^{(1)}(\beta, t) - 1) \\ B(\beta, t) &= \begin{pmatrix} \kappa_1(t)\beta_1(t, u) \\ \kappa_2(t)\beta_2(t, u) + \lambda_2(t)(\phi_J^{(2)}(\beta, t) - 1) + \frac{1}{2}B_1(\beta, t) \\ \kappa_3(t)\beta_3(t, u) \end{pmatrix} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} B_1(\beta, t) &= \beta_1(t, u)(\beta_1(t, u) + \beta_2(t, u)\rho_1(t)\sigma_2(t) + \beta_3(t, u)\rho_2(t)\sigma_3(t)) \\ &\quad + \beta_2(t, u)(\beta_1(t, u)\rho_1(t)\sigma_2(t) + \beta_2(t, u)\sigma_2^2(t) + \beta_3(t, u)\rho_1(t)\rho_2(t)\sigma_2(t)\sigma_3(t)) \\ &\quad + \beta_3(t, u)(\beta_1(t, u)\rho_2(t)\sigma_3(t) + \beta_2(t, u)\rho_1(t)\rho_2(t)\sigma_2(t)\sigma_3(t) + \beta_3(t, u)\rho_2^2(t)\sigma_3^2(t)) \end{aligned}$$

3 Pricing of Energy Commodity Derivatives

We have specified three types of mean-reverting jump-diffusion processes for modeling the energy commodity spot prices. We also showed how to compute the generalized transform function of the processes at any given time T. The prices of European type contingent claims on the spot price can now be obtained through the inversion of the transform functions. Let $\overline{X}_t$ be a state vector in R^n and $u \in C^n$, the generalized transform function is given by

$$\begin{aligned} \varphi(u, \overline{X}_t, t, T) &\equiv E^Q[e^{-r(T-t)} \exp(u \cdot \overline{X}_T) | \mathcal{F}_t] \\ &= \exp[\alpha(t, u) + \beta(t, u) \cdot \overline{X}_t] \end{aligned} \tag{3.1}$$

Let $G(v, X_t, Y_t, t, T; \overline{a}, \overline{b})$ denote the time-t price of a contingent claim with payoff $\exp(\overline{a} \cdot \overline{X}_T)$ when $\overline{b} \cdot \overline{X}_T \leq v$ is true at time T where $\overline{a}$, $\overline{b}$ are vectors in R^n and $v \in R^1$, then we have

$$\begin{aligned} G(v, \overline{X}_t, t, T; \overline{a}, \overline{b}) &= E^Q[e^{-r(T-t)} \exp(\overline{a} \cdot \overline{X}_T) \mathbf{1}_{\overline{b} \cdot \overline{X}_T \leq v} \mid \mathcal{F}_t] \\ &= \frac{\varphi(\overline{a}, \overline{X}_t, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\varphi(\overline{a} + iw\overline{b}, \overline{X}_t, t, T) e^{-iwv}]}{w} dw \end{aligned} \quad (3.2)$$

(See appendix for a formal proof.) To illustrate the results, we take some concrete examples of the three models and compute the price for several commonly traded energy commodity derivatives. Specifically, model 1a is a special case of model 1; model 2a is a special case of model 2; and model 3a is a special case of model 3. Closed-form solutions of the derivative securities (up to Fourier inversion) are provided whenever available.

3.1 Illustrative Models

The specific models we presented here are obtained by setting the parameters in the three general models to be constants. The jumps appear in the primary commodity price only and the jump sizes are distributed as independent exponential random variables in R^n. Therefore the transform function of the jump size takes the form of

$$\phi_J^j(\overline{c}, t) \equiv \prod_{k=1}^{n} \frac{1}{1 - \mu_j^k c_k} \quad (3.3)$$

3.1.1 Model 1a

Model 1a is a special case of (2.1) with all parameters being constants. The jumps are in X, the logarithm of the commodity spot price, only. The size of type-j jump ($j = 1$, 2) is exponentially distributed with mean μ_J^j. The transform function of the jump size is $\phi_J^j(c_1, c_2, t) \equiv \dfrac{1}{1 - \mu_J^j c_1}$ ($j = 1, 2$).

$$\begin{aligned} d \begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} \kappa_1(\theta_1 - X_t) \\ \kappa_2(\theta_2 - Y_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ \rho_1 \sigma_2 & \sqrt{1 - \rho_1^2} \sigma_2 \end{pmatrix} dW_t \\ &\quad + \sum_{i=1}^{2} \Delta Z_t^i \end{aligned} \quad (3.4)$$

The transform function φ_{1a} The closed-form solution of the transform function can be written out explicitly for this model.

$$\varphi_{1a}(u, X_t, Y_t, t, T) = \exp[\alpha(\tau) + \beta_1(\tau)X_t + \beta_2(\tau)Y_t] \tag{3.5}$$

where $\tau = T - t$. By solving the ordinary differential equations in (2.5) with all parameters being constants, we get the following,

$$\begin{aligned}
\beta_1(\tau, u_1) &= u_1 \exp(-\kappa_1\tau) \\
\beta_2(\tau, u_2) &= u_2 \exp(-\kappa_2\tau) \\
\alpha(\tau, u) &= -r\tau - \sum_{j=1}^{2} \frac{\lambda_J^j}{\kappa_1} \ln \frac{u_1\mu_J^j - 1}{u_1\mu_J^j \exp(-\kappa_1\tau) - 1} + \frac{a_1\sigma_1^2 u_1^2}{4\kappa_1} + \frac{a_2\sigma_2^2 u_2^2}{4\kappa_2} \\
&\quad + u_1\theta_1(1 - \exp(-\kappa_1\tau)) + u_2\theta_2(1 - \exp(-\kappa_2\tau)) \\
&\quad + \frac{u_1 u_2 \rho_1 \sigma_1 \sigma_2 (1 - \exp(-(\kappa_1 + \kappa_2)\tau))}{\kappa_1 + \kappa_2}
\end{aligned} \tag{3.6}$$

with $a_1 = 1 - \exp(-2\kappa_1\tau)$ and $a_2 = 1 - \exp(-2\kappa_2\tau)$.

3.1.2 Model 2a

Model 2a is a regime switching with the regime-jump only in the first commodity price. In the electricity markets, this is suitable for modeling the occasional price spikes in the electricity spot price caused by forced outage of the major plants or line contingency in transmission networks. For simplicity, we assume that there are no jumps within each regime.

$$\begin{aligned}
d\begin{pmatrix} X_t \\ Y_t \end{pmatrix} &= \begin{pmatrix} \kappa_1(\theta_1 - X_t) \\ \kappa_2(\theta_2 - Y_t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 & 0 \\ \rho_1\sigma_2 & \sqrt{1-\rho_1^2}\sigma_2 \end{pmatrix} dW_t \\
&\quad + \iota(U_{t-})dM_t
\end{aligned} \tag{3.7}$$

where W is an $\mathcal{F}_t$-adapted standard Brownian motion in $\Re^2$. U_t is the regime state process defined in (2.7). The size of regime-jumps is assumed to be distributed as exponential random variables and the transform functions of the regime-jump size are $\phi_\iota(c_1, c_2, t) \equiv \dfrac{1}{1 - \mu_\iota c_1}$ $(\iota = 0, 1)$ where $\mu_0 \geq 0$ and $\mu_1 \leq 0$.

The transform function φ_{2a} For this model the transform function φ_{2a} can not be solved in closed-form completely. We have

$$\begin{aligned}
\varphi_{2a}^0(x, y, t) &= \exp(\alpha_0(t) + \beta_1(t)x + \beta_2(t)y) \\
\varphi_{2a}^1(x, y, t) &= \exp(\alpha_1(t) + \beta_1(t)x + \beta_2(t)y)
\end{aligned} \tag{3.8}$$

where $\beta(t) \equiv \beta(t,u) \equiv (\beta_1(t,u), \beta_2(t,u))'$ has the closed-form solution of

$$\begin{aligned}\beta_1(\tau, u_1) &= u_1 \exp(-\kappa_1 \tau)\\ \beta_2(\tau, u_2) &= u_2 \exp(-\kappa_2 \tau)\end{aligned}$$

but $\alpha(t) \equiv \alpha(t,u) \equiv (\alpha_0(t,u), \alpha_1(t,u))'$ needs to be numerically computed from

$$\begin{aligned}\frac{d}{dt}\begin{pmatrix}\alpha_0(t)\\ \alpha_1(t)\end{pmatrix} &= -\begin{pmatrix} A_1(\beta(t),t) + \lambda^{(0)}[\dfrac{\exp(\alpha_1(t)-\alpha_0(t))}{1-\mu_0\beta_1(t,u_1)} - 1]\\ A_1(\beta(t),t) + \lambda^{(1)}[\dfrac{\exp(\alpha_0(t)-\alpha_1(t))}{1-\mu_1\beta_1(t,u_1)} - 1]\end{pmatrix}\\ \begin{pmatrix}\alpha_0(0,u)\\ \alpha_1(0,u)\end{pmatrix} &= \begin{pmatrix}0\\0\end{pmatrix}\end{aligned}$$

with

$$A_1(\beta(t),t) = -r + \sum_{i=1}^{2}[\kappa_i\theta_i\beta_i + \frac{1}{2}\sigma_i^2\beta_i^2] - \rho_1\sigma_1\sigma_2\beta_1\beta_2$$

3.1.3 Model 3a

Model 3a is a stochastic volatility model in which the type-1 jumps are simultaneous jumps in the commodity spot price and volatility, and the type-2 jumps are in the commodity spot price only. All parameters are constants.

$$\begin{aligned} d\begin{pmatrix}X_t\\ V_t\\ Y_t\end{pmatrix} &= \begin{pmatrix}\kappa_1(\theta_1 - X_t)\\ \kappa_V(\theta_V - V_t)\\ \kappa_2(\theta_2 - Y_t)\end{pmatrix} dt + \begin{pmatrix}\sqrt{V_t} & 0 & 0\\ \rho_1\sigma_2\sqrt{V_t} & \sqrt{1-\rho_1^2}\sigma_2\sqrt{V_t} & 0\\ \rho_2\sigma_3\sqrt{V_t} & 0 & \sigma_3\end{pmatrix} dW_t\\ &\quad + \sum_{i=1}^{2}\Delta Z_t^i \end{aligned} \tag{3.9}$$

where W is an $\mathcal{F}_t$-adapted standard Brownian motion in $\Re^3$; Z^i $(i = 1,2)$ is a compound Poisson process in $\Re^3$. The Poisson arrival intensity functions are $\lambda^1(X_t, V_t, Y_t, t) = \lambda_1$ and $\lambda^2(X_t, V_t, Y_t, t) = \lambda_2 V_t$. The transform functions of the jump size are

$$\begin{aligned}\phi_J^1(c_1, c_2, c_3, t) &\equiv \frac{1}{(1-\mu_1^1 c_1)(1-\mu_1^2 c_2)}\\ \phi_J^2(c_1, c_2, c_3, t) &\equiv \frac{1}{1-\mu_2^1 c_1}\end{aligned}$$

where μ_J^k is the mean size of the type-J $(J = 1,2)$ jump in factor k $(k = 1,2)$.

The transform function φ_{3a} ¿From the previous section, we know φ_{3a} is of form

$$\varphi_{3a}(u, X_t, V_t, Y_t, t, T) = \exp(\alpha(t,u) + \beta_1(t,u)X_T + \beta_2(t,u)V_T + \beta_3(t,u)Y_T)$$

Similar to model 2a, the transform function φ_{3a} does not have a closed-form solution. We numerically solve for both $\alpha(t,u)$ and $\beta(t,u) \equiv [\beta_1(t,u), \beta_2(t,u), \beta_3(t,u)]'$ from the following ordinary differential equations (ODEs)

$$\begin{aligned} \frac{d}{dt}\beta(t,u) + B(\beta(t,u),t) = 0, &\qquad \beta(0,u) = u \\ \frac{d}{dt}\alpha(t,u) + A(\beta(t,u),t) = 0, &\qquad \alpha(0,u) = 0 \end{aligned} \tag{3.10}$$

with

$$A(\beta,t) = -r + \sum_{i=1}^{3}\kappa_i\theta_i\beta_i(t,u) + \tfrac{1}{2}\beta_3^2(t)\sigma_3^2 + \lambda_1[\frac{1}{(1-\mu_1^1\beta_1(t,u))(1-\mu_1^2\beta_2(t,u))} - 1]$$

$$B(\beta,t) = \begin{pmatrix} \kappa_1\beta_1(t,u) + \lambda_{21}(\dfrac{1}{1-\mu_2^1\beta_1(t,u)} - 1) \\ \kappa_2\beta_2(t,u) + \lambda_{22}(\dfrac{1}{1-\mu_2^1\beta_1(t,u)} - 1) + \tfrac{1}{2}B_1(\beta,t) \\ \kappa_3\beta_3(t,u) \end{pmatrix} \tag{3.11}$$

and

$$\begin{aligned} B_1(\beta,t) &= \beta_1(t,u)(\beta_1(t,u) + \beta_2(t,u)\rho_1\sigma_2 + \beta_3(t,u)\rho_2\sigma_3) \\ &\quad + \beta_2(t,u)(\beta_1(t,u)\rho_1\sigma_2 + \beta_2(t,u)\sigma_2^2 + \beta_3(t,u)\rho_1\rho_2\sigma_2\sigma_3) \\ &\quad + \beta_3(t,u)(\beta_1(t,u)\rho_2\sigma_3 + \beta_2(t,u)\rho_1\rho_2\sigma_2\sigma_3 + \beta_3(t,u)\rho_2^2\sigma_3^2) \end{aligned}$$

3.2 Energy Commodity Derivatives

In this subsection, we provide the pricing formulae for the futures, calls, spark spread options, and locational spread options. We use parameters given in Table (??) to explicitly compute the derivative prices. Comparisons of the derivative prices in different models will be shown.

3.2.1 Futures/Forward Price

A futures (forward) contract promising to deliver one unit of commodity S^i at a future time T for a price of F has the following payoff at time T

$$\text{Payoff} = S_T^i - F$$

Since there is no initial payment is required to enter into a futures contract, the futures price F is given by

$$F(S_t^i, t, T) = E^Q[S_T^i \mid \mathcal{F}_t]$$

Rewrite the above expression as

$$\begin{aligned} F(S_t^i, t, T) &= E^Q[S_T^i \mid \mathcal{F}_t] \\ &= e^{r\tau} E^Q[e^{-r\tau} \cdot \exp(X_T^i) \mid \mathcal{F}_t] \end{aligned}$$

We therefore have

$$F(S_t^i, t, T) = e^{r\tau} \cdot \varphi(\overline{e}_i^T, \overline{X}_t, \tau) \tag{3.12}$$

where $\tau = T - t$; $\varphi(u, \overline{X}_t, \tau)$ is the transform function in (3.1); $\overline{e}_i$ is the vector with i^{th} component being 1 and other components all being 0.

Futures price (model 1a) Recall the transform function φ_{1a} is given by (3.5) and (3.6). By setting $u = [1, 0]'$ in φ_{1a}, we get

$$\varphi_{1a}([1, 0]', X_t, Y_t, \tau) = \exp[X_t \exp(-\kappa_1 \tau) - r\tau + \frac{a_1 \sigma_1^2}{4\kappa_1} + \theta_1(1 - \exp(-\kappa_1 \tau)) - j(\tau)]$$

where $\tau = T - t$, $X_t = \ln(S_t)$, $a_1 = 1 - \exp(-2\kappa_1 \tau)$ and $j(\tau) = \sum_{j=1}^{2} \frac{\lambda_J^j}{\kappa_1} \ln \frac{\mu_J^j - 1}{\mu_J^j \exp(-\kappa_1 \tau) - 1}$. Therefore by (3.12), we have the following proposition.

Proposition 1 *In* **Model 1a**, *the futures price of commodity S_t at time t with delivery time T is*

$$\begin{aligned} F(S_t, t, T) &= e^{r\tau} \cdot \varphi_{1a}([1, 0]', X_t, Y_t, \tau) \\ &= \exp[X_t \exp(-\kappa_1 \tau) + \frac{a_1 \sigma_1^2}{4\kappa_1} + \theta_1(1 - \exp(-\kappa_1 \tau)) - j(\tau)] \end{aligned} \tag{3.13}$$

where $\tau = T - t$, $X_t = \ln(S_t)$, $a_1 = 1 - \exp(-2\kappa_1 \tau)$ and $j(\tau) = \sum_{j=1}^{2} \frac{\lambda_J^j}{\kappa_1} \ln \frac{\mu_J^j - 1}{\mu_J^j \exp(-\kappa_1 \tau) - 1}$.

Note that the futures price in this model is simply the futures price in the mean-reversion model without any jumps but scaled up by a factor of $\exp(-j(\tau))$. If we interpret the spikes in commodity spot price as upwards jumps followed shortly by downwards jumps of similar sizes, then over a long time horizon we can say that both the frequencies and the sizes of the upwards jumps and the downwards jumps are roughly the same, i.e. $\lambda_J^1 = \lambda_J^2$ and $\mu_J^1 \approx -\mu_J^2$. One might intuitively think that the up and down jumps would offset each other's effect in the futures price. What (3.13) tells us is that this intuition is not quite right and indeed, in the case where $\lambda_J^1 = \lambda_J^2$ and $\mu_J^1 = -\mu_J^2$, the futures price is definitely higher than that corresponding to the no-jump case.

Futures price (model 2a) The futures price in model 2a is

$$F(S_t, t, T) = e^{r\tau} \cdot \varphi_{2a}^i([1, 0]', X_t, Y_t, \tau) \quad (i = 0, 1)$$

where i is the Markov regime state variable; $\tau = T - t$ and the transform functions φ_{2a}^i are computed in (3.8).

Futures price (model 3a) The futures price in model 3a is

$$F(S_t, t, T) = e^{r\tau} \cdot \varphi_{3a}([1, 0, 0]', X_t, V_t, Y_t, \tau)$$

where $\tau = T - t$ and the transform function φ_{3a} is given in (3.1.3).

Forward curves Using the parameters in Table (??) for modeling the electricity spot price in the Eastern region of United States (Cinergy to be specific), we obtain forward curves for Cinergy under each of the three illustrative models. The second jointly specified factor process is the spot price of natural gas at Henry Hub. For the initial values of $S_e = 24.63$, $S_g = 2.105$, $V = 0.5$, $U = 0$ and $r = 4\%$, Figure (??) illustrates the three forward curves for electricity.

3.2.2 Call Option

A "plain vanilla" European call option on commodity S^i with strike price K has the payoff of

$$C(S_T^i, K, T, T) = \max(S_T^i - K, 0)$$

at maturity time T . The price of the call option is given by

$$\begin{aligned} C(S_t^i, K, t, T) &= E^Q[e^{-r(T-t)} \max(S_T^i - K, 0) \mid \mathcal{F}_t] \\ &= E^Q[e^{-r\tau} \exp(X_T^i) 1_{X_T^i \geq \ln K} \mid \mathcal{F}_t] - K \cdot E^Q[e^{-r\tau} 1_{X_T^i \geq \ln K} \mid \mathcal{F}_t] \\ &= G_1 - K \cdot G_2 \end{aligned} \tag{3.14}$$

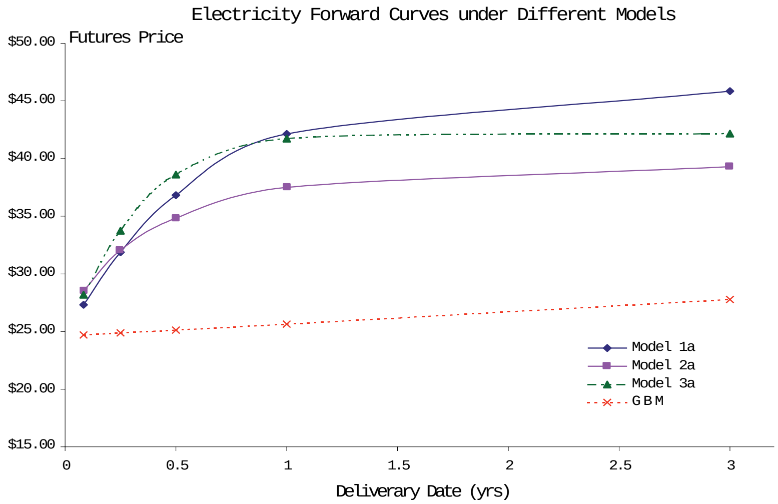

Figure 4: Forward Curves under Different Models

where $\tau = T - t$ and G_1, G_2 are obtained by setting $\{a = \overline{e}_i\ ,\ b = -\overline{e}_i,\ v = -\ln K\}$ and $\{a = \overline{0}\ ,\ b = -\overline{e}_i,\ v = -\ln K\}$ in (3.2), respectively.

$$\begin{aligned} G_1 &= E^Q[e^{-r\tau}\exp(X_T)1_{X_T \geq \ln K} \mid \mathcal{F}_t] \\ &= \frac{F_t^i e^{-r\tau}}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\varphi([1-w\cdot i, 0], \overline{X}_t, \tau)\exp(i\cdot w\ln K)]}{w}dw \\ &= F_t^i e^{-r\tau}(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\varphi([1-w\cdot i, 0], \overline{X}_t, \tau)\exp(r\tau + i\cdot w\ln K)]}{wF_t^i}dw) \quad (3.15) \end{aligned}$$

where $F_t^i = e^{r\tau}\cdot\varphi(\overline{e}_i^T, \overline{X}_t, \tau)$ is the time-t forward price of commodity S^i with delivery time T.

$$\begin{aligned} G_2 &= E^Q[e^{-r\tau}1_{X_T^i \geq \ln K} \mid \mathcal{F}_t] \\ &= \frac{\varphi(\overline{0}, \overline{X}_t, t, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\varphi(i\cdot w\overline{e}_i, \overline{X}_t, t, T)e^{-i\cdot w\ln v}]}{w}dw \\ &= e^{-r\tau}\left(\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\varphi(i\cdot w\overline{e}_i, \overline{X}_t, t, T)e^{r\cdot\tau - i\cdot w\ln v}]}{w}dw\right) \quad (3.16) \end{aligned}$$

Call option price Substituting φ_{1a}, φ_{2a}, and φ_{3a} into (3.15) and (3.16) we have the call option price given by (3.14) under **Model 1a**, **2a** and **3a**, respectively.

Volatility smile Figure (5) plots the call option values with different maturity time under different models. The call values under a Geometric Brownian motion (GBM) model are also

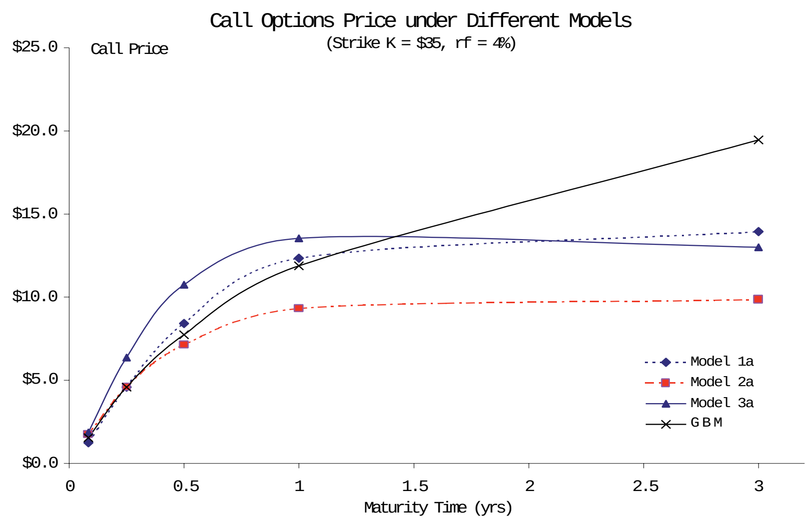

Figure 5: Call Options Price under Different Models

plotted for comparison purpose. Note that, as maturity time increases, the mean-reversion effects in all three models cause the value of a call option to converge to a long-term value which is different from the spot price of the underlying. Figure (6) illustrates the implied volatility curves under the three illustrative models for the parameter sets given in Table (3.4).

3.2.3 Cross Commodity Spread Option

In energy commodity markets, cross commodity derivatives play crucial roles in risk management. Crack spread options in crude oil markets as well as the spark spread and locational spread options in electricity markets are good examples. Deng, Johnson and Sogomonian (1998) illustrated how the spark spread options, which are derivatives on electricity and the fossil fuels used to generate electricity, can have various applications in risk management for utility companies and power marketers. Moreover, such options are essential in asset valuation for fossil fuel electricity generation plants. A *spark spread call (SSC)* option pays off the positive part of the difference between the electricity spot price and the generating fuel cost. It has the following payoff function at the time of maturity,

$$SSC(S_T^e, S_T^g, H, T) = \max(S_T^e - H \cdot S_T^g, 0)$$

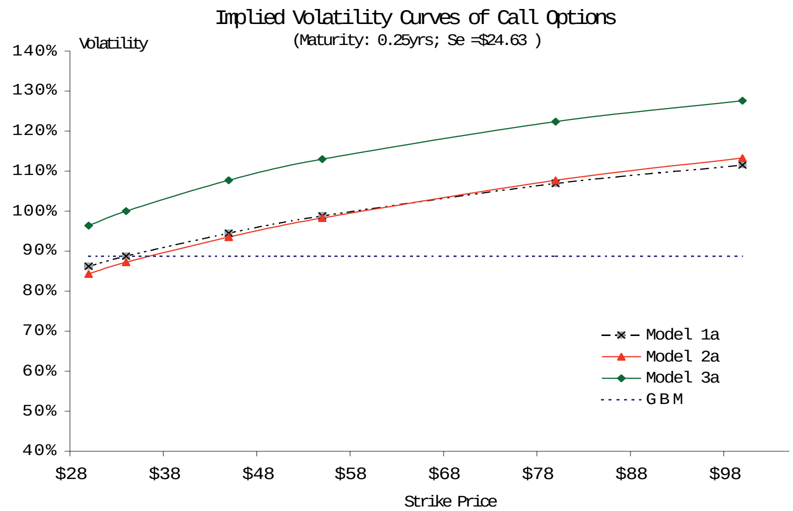

Figure 6: Volatility-Smile under Different Models

where S_T^e and S_T^g are the price of the electricity and the generating fuel, respectively; the constant H, termed *strike heat rate*, represents the number of units of generating fuel contracted to produce one unit of electricity.

Another kind of cross commodity option called locational spread option is also introduced in Deng, Johnson and Sogomonian (1998). A locational spread option pays off the positive part of the commodity price difference at two different delivery points. In the context of electricity markets, they serve the purposes of hedging the transmission risk and they can also be used to value transmission expansion projects as shown in Deng, Johnson and Sogomonian (1998). The time T payoff of a *locational spread call* option is

$$LSC(S_T^a, S_T^b, L, T) = \max(S_T^b - L \cdot S_T^a, 0)$$

where S_T^a and S_T^b are the time-T commodity prices at location a and b, and the constant L is a loss factor reflecting the fact that there might be a transportation / transmission loss associated with shipping one unit of the commodity from location a to b.

Observing the similar payoff structures of the above two spread options, we define a general *cross-commodity spread call* option as an option with the following payoff at maturity time T,

$$CSC(S_T^1, S_T^2, K, T, T) = \max(S_T^1 - K \cdot S_T^2, 0)$$

where K is a scaling constant associated with the spot price of commodity 2. The interpretation for K is different for different instruments, for example, K is equal to the heat rate H for a spark spread option and it is the loss factor L for a locational spread option.

The value of a European cross-commodity spread call option on S_t^1 and S_t^2 at time t is given by

$$\begin{aligned} CSC(S_t^1, S_t^2, K, t) &= E^Q[e^{-r(T-t)} \max(S_T^1 - K \cdot S_T^2, 0) \mid \mathcal{F}_t] \\ &= E^Q[e^{-r\tau} \exp(X_T^1) 1_{S_T^1 - K \cdot S_T^2 \geq 0} \mid \mathcal{F}_t] - E^Q[e^{-r\tau} K \exp(X_T^2) 1_{S_T^1 - K \cdot S_T^2 \geq 0} \mid \mathcal{F}_t] \\ &= G_1 - G_2 \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} G_1 &= G(0, \ln S_t^1, \ln(K \cdot S_t^2), t, T; [1, 0, \cdots, 0]', [-1, 1, 0, \cdots, 0]') \\ G_2 &= G(0, \ln S_t^1, \ln(K \cdot S_t^2), t, T; [0, 1, 0, \cdots, 0]', [-1, 1, 0, \cdots, 0]') \end{aligned} \tag{3.18}$$

and recall that

$$G(v, \overline{X}_t, t, T; \overline{a}, \overline{b}) = \frac{\varphi(\overline{a}, \overline{X}_t, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\varphi(\overline{a} + iw\overline{b}, \overline{X}_t, t, T) e^{-iwv}]}{w} dw$$

Cross commodity spread call option price under each of the three models are obtained by substituting φ_{1a}, φ_{2a}, and φ_{3a} into (3.17) and (3.18).

The spark spread call option value with strike heat rate $H = 9.5$ for different maturity time is shown in figure (7).

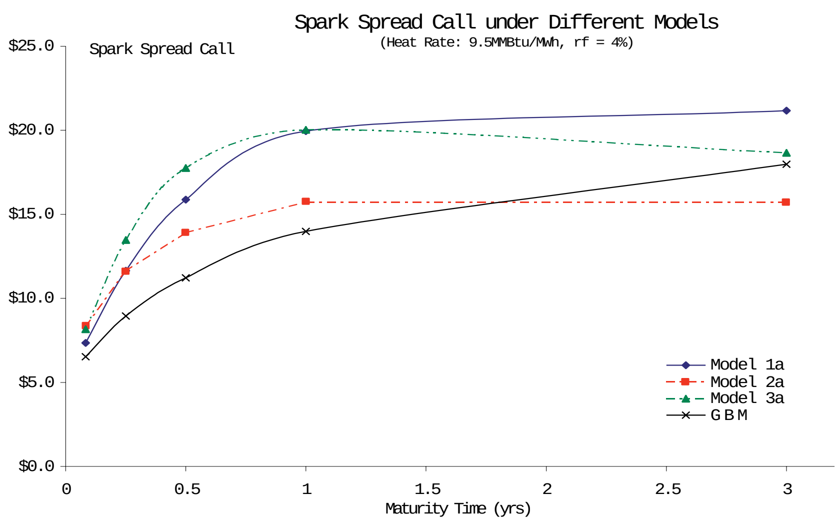

Figure 7: Spark Spread Call Price under Different Models

3.3 Comparative Statics

Let X_t^i denote $\ln S_t^i$ and $\overline{e}_i$ be the vector with the i^{th} component being 1 and all other components being 0. By setting $a = \overline{0}$, $b = \overline{e}_i$ in (3.2), we obtain the cumulative distribution function $CF_{S_T^i}(v)$ of S_T^i under the risk-neutral measure at time t given the parameters of the process and S_t^i:

$$
\begin{aligned}
CF_{S_T^i}(v) &\equiv \Pr[S_T^i \le v] \\
&= \Pr[X_T^i \le \ln v] \\
&= e^{r(T-t)} E^Q[e^{-r(T-t)} \exp(\overline{0} \cdot \overline{X}_T) \mathbf{1}_{\overline{e}_i \cdot \overline{X}_T \le \ln v} \mid \mathcal{F}_t] \\
&= e^{r(T-t)} \left(\frac{\varphi(\overline{0}, \overline{X}_t, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\varphi(i \cdot w\overline{e}_i, \overline{X}_t, t, T) e^{-i \cdot w \ln v}]}{w} dw \right) \\
&= \frac{1}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\varphi(i \cdot w\overline{e}_i, \overline{X}_t, t, T) e^{r(T-t) - i \cdot w \ln v}]}{w} dw \qquad (3.19)
\end{aligned}
$$

Whenever the closed-form solution of the transform function φ is available, we can analytically investigate the stochastic monotonicity (in the first order stochastic dominance sense) of S_T^i with respect to the model parameters and initial conditions from (3.19). The comparative statics results of derivative security price with respect to the process parameters can therefore be established for those derivative securities having monotonic payoff functions in the underlying price S_T^i. We illustrate this point through the following proposition and corollary.

Proposition 2 *In* **Model 1a**, *under proper conditions for λ_J^j and μ_J^j, the terminal price S_T^1 conditioning on S_t^1 is stochastically increasing in the first order stochastic dominance sense in θ_i, S_t^1; and S_t^1 is stochastically increasing in λ_1 , μ_1 but stochastically decreasing in λ_2, $-\mu_2$.*

Proof. See Appendix.

Since the payoff functions of the futures and call options are increasing functions in underlying price S_t^i, we have the following corollary.

Corollary 1 *In* **Model 1a**, *under proper conditions for λ_J^j and μ_J^j the futures and call option prices of S^1 are increasing functions of θ_i, σ_i, S_t^1, λ_1, and μ_1; but decreasing functions of λ_2 and $-\mu_2$.*

Proof. By Propositon 1, the previous proposition, and an equivalent definition of the First Order Stochastic Dominance which states that if X first order stochastically dominates Y, then $E[f(X)] \geq E[f(Y)]$ for any increasing function $f(x)$.

3.4 Parameter Estimation

In this subsection, we illustrate how we can estimate the parameters for the models we considered in Section 3.1 using the electricity price data. For illustration purpose, we pick **Model 1a** and derive the moment conditions from the transform function for the unconditional distribution of the underlying price return. We assume the risk premium associated with the factor X to be proportional to X, i.e. the risk premium is of form $\xi_X \cdot X$. For simplicity, we further assume the risk premia associated with the jumps are zero. Therefore the price processes under the true measure is of the same form. In particular,

$$\begin{aligned} \kappa_i &= \kappa_i^* + \xi_i \\ \theta_i &= \frac{\kappa_i^* \cdot \theta_i^*}{\kappa_i^* + \xi_i} \\ \lambda_j &= \lambda_j^* \\ \mu_j &= \mu_j^* \end{aligned}$$

We then use the electricity and natural gas spot and futures price series to get the estimates for the parameters under the true measure and the risk premia by matching moment conditions and the futures prices. The following proposition provides the mean, variance and skewness of the logarithm of the electricity price in **Model 1a**.

Proposition 3 *In* **Model 1a** *where* $X_t = \ln S_t^e$, *let* $X_\infty \equiv \lim_{t\to\infty} X_t$ *denote the unconditional distribution of* X_t. *If* $E[|X_\infty|^n] < \infty$, *then the mean, variance and skewness of* X_∞ *are*

$$\begin{aligned} mean &= \theta_1 + \frac{\lambda_1\mu_1 + \lambda_2\mu_2}{\kappa_1} \\ variance &= \frac{\sigma_1^2}{2\kappa_1} + \frac{\lambda_1\mu_1^2 + \lambda_2\mu_2^2}{\kappa_1} \\ skewness &= \frac{4\sqrt{2\kappa_1}(\lambda_1\mu_1^3 + \lambda_2\mu_2^3)}{(\sigma_1^2 + 2\lambda_1\mu_1^2 + 2\lambda_2\mu_2^2)^{\frac{3}{2}}} \end{aligned}$$

Proof. In the transform function φ_{1a} as defined by (3.5) and (3.6), let $u = i \cdot w$ and $t \to \infty$, we obtain the characteristic function of X_∞ to be

$$\Phi_{X_\infty}(w) = \exp[i \cdot w\theta_1 - \frac{w^2\sigma_1^2}{4\kappa_1} - \sum_{j=1}^{2} \frac{\lambda_J^j}{\kappa_1} \ln(1 - i \cdot w\mu_J^j)]$$

If $E[|X_\infty|^n] < \infty$, then the n^{th} moment of X_∞ is given by

$$E[X_\infty^n] = (-i)^n \left. \frac{d^n}{dw^n} \Phi_{X_\infty}(w) \right|_{w=0}$$

In particular,

$$\begin{aligned} E[X_\infty] &= (-i) \left. \frac{d}{dw} \Phi_{X_\infty}(w) \right|_{w=0} \\ &= \theta_1 + \frac{\lambda_1\mu_1 + \lambda_2\mu_2}{\kappa_1} \end{aligned}$$

The formulae for variance and skewness are obtained in the same fashion. ∎

As shown in the proof of the Proposition (3), we can obtain as many moment conditions as we desire for X_∞ from the characteristic function of X_∞ for the estimation of the model parameters of **Model 1a**. The parameters in **Model 2a** and **Model 3a** are obtained by minimizing the mean squared errors of the traded options prices.

4 Implication to Capacity Valuation and the Value of Investment Opportunity

In this section, we first illustrate how the energy commodity derivative securities can be utilized to value installed capacity or real assets. Next we examine the value of investment opportunity and the timing of investment when the value process associated with an investment has random up and down jumps in it.

4.1 Real Options Valuation of Capacity

For the ease of exposition, we illustrate how to use derivative securities to value real assets through presenting examples from the electric power industry. This subsection follows the methodology proposed in Deng, Johnson and Sogomonian (1998) but here the valuation is based on assumptions about electricity spot price process instead of futures price process.

	Model 1a	Model 2a	Model 3a
κ_1	1.70	1.37	2.17
κ_2	1.80	1.80	3.50
κ_3	N/A	N/A	1.80
θ_1	3.40	3.30	3.20
θ_2	0.87	0.87	0.85
θ_3	N/A	N/A	0.87
σ_1	0.74	0.80	N/A
σ_2	0.34	0.34	0.80
σ_3	N/A	N/A	0.54
ρ_1	0.20	0.20	0.25
ρ_2	N/A	N/A	0.20
λ_1	6.08	6.42	6.43
μ_{11}	0.19	0.26	0.23
μ_{12}	N/A	N/A	0.22
λ_2	7.00	8.20	5.00
μ_{21}	-0.11	-0.20	-0.14

Table 1: Parameters for the Illustrative Models

Take a fossil fuel electric power generating plant, the spread between the price of electricity and the fuel that is used to generate it determines the economic value of the generation asset that can be used to transform the fuel into electricity. The amount of fuel that a particular generation asset requires to generate a given amount of electricity depends on the asset's efficiency. This efficiency is summarized by the asset's **operating heat rate**, which is defined as the number of British thermal units (Btus) of the input fuel (measured in millions) required to generate one megawatt hour (MWh) of electricity. Thus the lower the operating heat rate, the more efficient the facility. The right to operate a generation asset with operating heat rate H that uses generating fuel g is clearly given by the value of a spark spread option with "strike" heat rate H written on generating fuel g because they yield the same payoff. Similarly, the value of a transmission asset that connects location 1 to location 2 is equal to the sum of the value of the locational spread option to buy electricity at location 1 and sell it at location 2 and the value of the option to buy electricity at location 2 and sell it a location 1 (in both cases, less the appropriate transmission cost). This equivalence between the value of appropriately defined spark and locational spread options and the right to operate a generation or a transmission asset can be easily used to value such assets.

We demonstrate this approach by developing a simple spark spread based model of the value of a fossil fuel generation asset. Once established, we fit the model and use it to generate estimates of the value of several gas-fired plants that have recently been sold. The accuracy of the model is then evaluated by comparing the estimates constructed to the prices at which the assets were actually sold.

In the analysis we make the following simplifying assumptions about the operating characteristics of the generation assets under consideration:

Assumption 1 Ramp-ups and ramp-downs of the facility can be done with very little advance notice.

Assumption 2 The facility's operation (e.g. start-up/shutdown costs) and maintenance costs are constant.

These assumptions are reasonable, since for a typical gas turbine combined cycle cogeneration plant the response time (ramp up/down) is several hours and the variable costs (e.g. operation and maintenance) are generally stable over time.

To construct a spark spread based estimate of the value of a generation asset, we estimate the value of the right to operate the asset over its remaining useful life. This value can be found by integrating the value of the spark spread options over the remaining life of the asset. Specifically,

Definition 1 *Let one unit of the* **time-t capacity right** *of a fossil fuel fired electric power plant represent the right to convert K_H units of generating fuel into one unit of electricity by using the plant at time t, where K_H is the plant's operating heat rate.*

The payoff of one unit of time-t capacity right is $\max(S_E^t - K_H S_G^t, 0)$, where S_E^t and S_G^t are the spot prices of electricity and generating fuel at time t, respectively. Denote the value of one unit of the time-t capacity right by $u(t)$.

For a natural gas fired power plant, the value of $u(t)$ is given by the corresponding spark spread call option on electricity and natural gas with a strike heat rate of K_H. However, for a coal-fired power plant, it often has a long-term coal supply contract which guarantees the supply of coal at a predetermined price c. Therefore the payoff of one unit of time-t capacity right for a coal plant degenerates to that of a call option with strike price $K_H \cdot c$. In this case $u(t)$ is equal to the value of a call option.

Definition 2 *Denote the* **virtual value** *of one unit of capacity of a fossil fuel power plant by V. Then V is equal to one unit of the plant's time-t capacity right over the remaining life $[0,T]$ of the power plant, i.e.* $V = \int_0^T u(t)dt$.

A capacity value curve using the initial price data at Palo Verde on 10/15/97 and the parameters obtained by fitting **Model 1a** to electricity price at Pale Verde and natural gas price at Henry Hub is shown in Figure (8). For comparison purpose, Figure (8) also plots

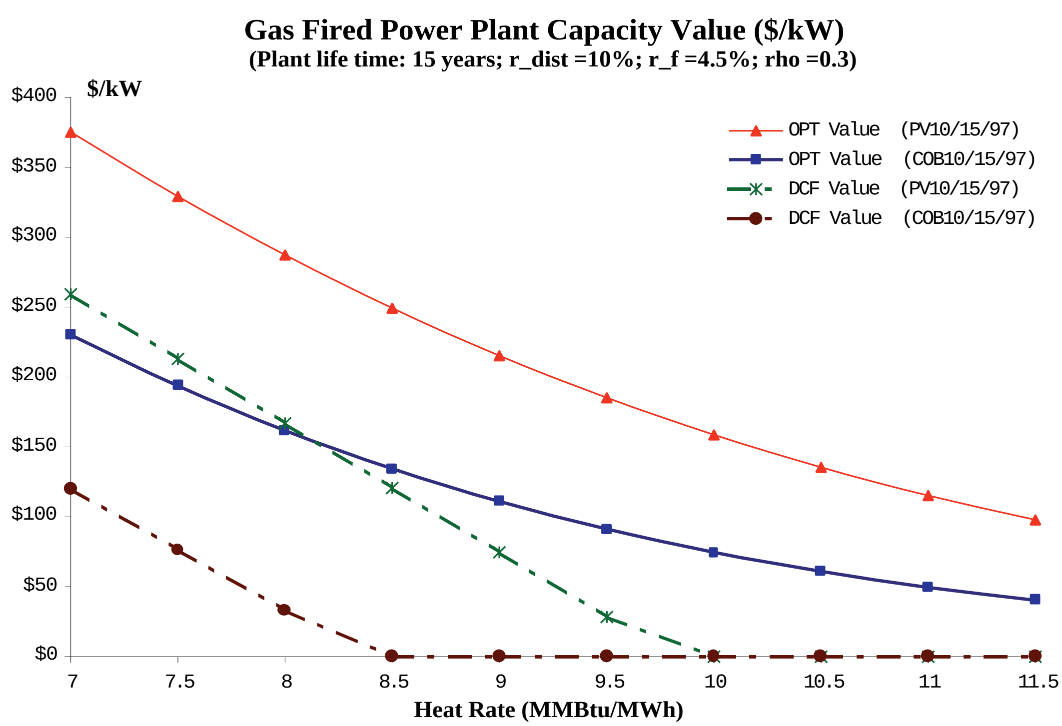

Figure 8: Value of Capacity under Different Models

the two capacity value curves obtained by using the discounted cash flow method and by assuming electricity spot price follows geometric Brownian motion (GBM), respectively. At the heat rate level of 9500, the capacity value of a natural gas power plant is around $200/kW under **Model 1a**. The discounted cash flow method predicts a value of $29/kW. To put things in perspective, we take a look at the four gas-fired power plants which Southern California Edison recently sold to Houston Industries. Unfortunately, not all of the individual plant dollar investments have been made public yet. As a proxy we therefore use the total investment made by Houston Industries (\$237 million to purchase four plants - Coolwater, Ellwood, Etiwanda and Mandalay), divided by the total number of megawatts (MW) of capacity (2172MW) to get approximately \$110,000 per MW (or \$110/kW) of capacity. However, the Coolwater Plant[2], in Daggett, California, is the most efficient (with an average heat rate of 9,500) of the four plants in the package and thus should have a higher value per MW. We therefore assume that the implied market value for Coolwater could range from \$110,000 to \$220,000 per MW, or equivalently, \$110/kW to \$220/kW. Our real options based estimate of the capacity value explains the market valuation much better than does the discounted cash flow estimate.

4.2 Value of Investment Opportunity and When to Invest

In the previous subsection, we obtain the value of installed capacity by viewing the capacity as a real option whose payoff structure can be replicated by a bunch of financial options. We consider now the following related questions: given the opportunity to incur a sunk investment cost K to install the capacity and realize the value V, what is the value of such an investment opportunity and when is the best time to realize that option. The existing literatures suggest that without jumps in the investment value process the investment opportunity value depends on the convenience yield and volatility of the investment value process and a firm should wait to invest until the value V rises to a threshold level V^*. When the spot price process of a commodity exhibits significant jumps, it is conceivable that the value of investment in its production facility should also jump up and down from time to time. In this subsection, we consider that the value of investment, V_t, evolves according to a regime-switching process which alternates back and forth between "high" and "low" states through jumps of random size. Such a regime-switching setting is appropriate, for example, in the current deregulated electricity industry. When the spot price of electricity is unusually high, firms are attracted to invest in building new plants. This may result in excess capacity for the subsequent years causing the value process of investing in new capacity to drop into

[2]The Coolwater Plant is made up of four units. Two 256MW Combined Cycle Gas Turbines plus a steam turbine; and two conventional turbines with capacity 65MW and 81MW each. Some re-power work has been done on the larger units.

"low" state. The low state will prevail until events such as decommissioning of a nuclear plant or persistent load growth occur which cause the value process to jump back into "high" state.

Specifically, let $X_t \equiv \ln V_t$, we model X_t as the following process

$$dX_t = (r - \delta - \frac{\sigma^2}{2})dt + \sigma dB_t + \iota(U_{t-})dM_t$$

where B_t is a standard Brownian motion in R^1; r is the risk free interest rate; δ is the convenience yield of the installed capacity; U_t is the regime state variable defined in (2.7); $M(t)$ is the compensated continuous-time Markov chain defined in (2.8). $\iota(0)$ and $-\iota(1)$ are the random variables representing the random jump size associated with the regime switching. They have distribution functions of $\upsilon_0(z)$ and $\upsilon_1(z)$.

Let $F^i(X_t)$ denote the value of investment opportunity when the regime state is i ($i = 0, 1$). By applying the Hamilton-Bellman-Jacobi equation in each state i, we have

$$\begin{cases} (r - \delta - \frac{\sigma^2}{2})F_0'(x) + \frac{1}{2}\sigma^2 F_0''(x) + \lambda_0 \int_{-\infty}^{+\infty}[F_1(x+z) - F_0(x)]d\upsilon_0(z) = rF_0(x) \\ (r - \delta - \frac{\sigma^2}{2})F_1'(x) + \frac{1}{2}\sigma^2 F_1''(x) + \lambda_1 \int_{-\infty}^{+\infty}[F_0(x+z) - F_1(x)]d\upsilon_1(z) = rF_1(x) \end{cases} \tag{4.1}$$

We conjecture the solutions have the following form

$$F_i(x) = \exp(\alpha_i + \beta x) \quad i = 0, 1$$

By further assuming $\iota(0)$ and $-\iota(1)$ are exponential random variables with mean μ_0 and μ_1, respectively, we simplify (4.1) to

$$\begin{cases} (r - \delta - \frac{\sigma^2}{2})\beta + \frac{1}{2}\sigma^2\beta^2 + \lambda_0(\frac{e^{\alpha_1 - \alpha_0}}{1 - \mu_0\beta} - 1) = r \\ (r - \delta - \frac{\sigma^2}{2})\beta + \frac{1}{2}\sigma^2\beta^2 + \lambda_1(\frac{e^{\alpha_0 - \alpha_1}}{1 + \mu_1\beta} - 1) = r \end{cases} \tag{4.2}$$

Intuitively, a firm would only exercise the investment option in the "high" state. In the "low" state a firm is always better off by waiting since it knows the value of investment will eventually jump up. Therefore we have the value matching and smooth pasting conditions in the "high" state $i = 1$ only,

$$F_1(x^*) = \exp(\alpha_1 + \beta x^*) = \exp(x^*) - K \tag{4.3}$$

$$F_1'(x^*) = \beta \exp(\alpha_1 + \beta x^*) = \exp(x^*) \tag{4.4}$$

¿From (4.2), (4.3) and (4.4) we can numerically solve for $(\alpha_0, \alpha_1, \beta, x^*)$. In particular, V^*, the threshold level to trigger investment is given by

$$V^* = \exp(x^*) = \frac{\beta}{\beta - 1}K \tag{4.5}$$

In what follows we set the investment cost K equal to 1, $r = 4\%$, $\delta = 5\%$, $\sigma = 0.4$, $\lambda_0 = 1.42$, $\lambda_1 = 2.95$, $\mu_0 = 8\%$, and $\mu_1 = 11\%$. Figure (9) plots $F_i(V)$ and the threshold V^* for investing for these parameters.

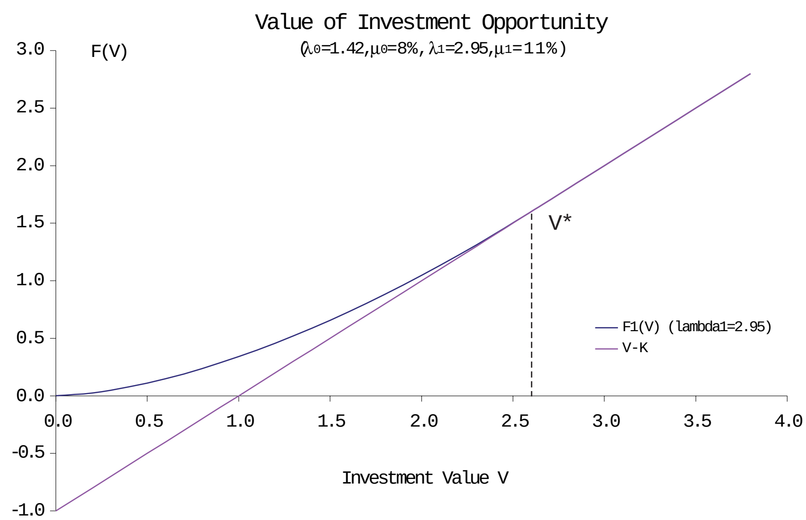

Figure 9: Value of Investment Opportunity

To contrast the above results with those from a jump-diffusion value process with two types of random up and down jumps, we introduce the following alternative investment value process

$$dX_t = (r - \delta - \frac{\sigma^2}{2})dt + \sigma dB_t + \sum_{i=0}^{1} \Delta Z_t^i$$

where $X_t = \ln V_t$, Z_t^i is a compound Poisson process in R^1 with a constant intensity of λ_i and jump size distributed as an exponential random variable with mean μ_i ($\mu_0 \geq 0$, $\mu_1 \leq 0$). The Hamilton-Bellman-Jacobi equation for the value function of the investment opportunity, $F(X_t)$, is given by

$$(r - \delta - \frac{\sigma^2}{2})F'(x) + \tfrac{1}{2}\sigma^2 F''(x) + \sum_{i=0}^{1} \lambda_i \int_{-\infty}^{+\infty} [F(x+z) - F(x)]dv_i(z) \quad = \quad rF(x) \tag{4.6}$$

Conjecture the solution to be of form $\exp(\alpha + \beta x)$, then (4.6) boils down to

$$(r - \delta - \frac{\sigma^2}{2})\beta + \tfrac{1}{2}\sigma^2\beta^2 + \sum_{i=0}^{1} \lambda_i(\frac{1}{1 - \mu_i\beta} - 1) \quad = \quad r \tag{4.7}$$

Along with the value matching condition (4.3) and the smooth pasting condition (4.4) we can solve for α , β and V^* . Indeed, V^* is again given by (4.5) but the β is solved from (4.7) instead.

In the case where the value process has random up and down jumps, firms are induced to invest only when the effect of downwards jumps dominates. Figure (10) shows the values

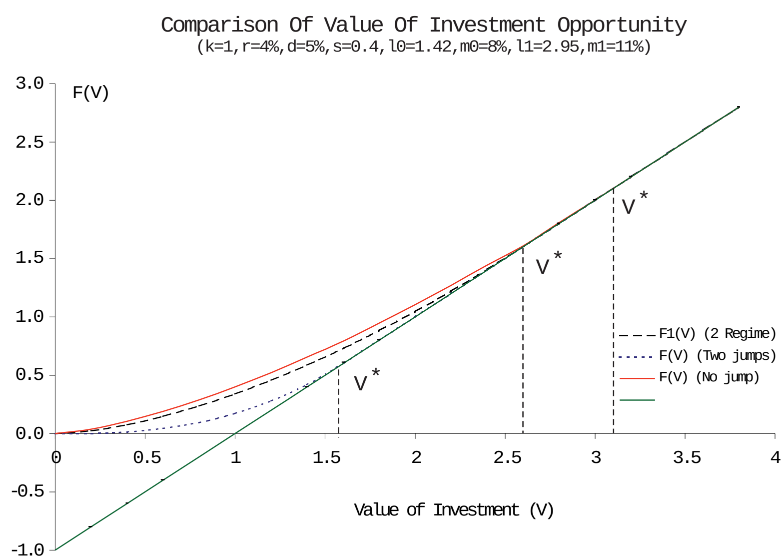

Figure 10: Comparison of Value to Invest under Different Models

of the opportunity to invest under the regime-switching, two types of random jumps and no-jump ($\lambda_0 = \lambda_1 = 0$) cases. We can see for the particular set of parameters the investment threshold is the highest in no-jump, lowest in random jump and regime-switching is in between.

5 Conclusion

In this paper, we propose three types of mean-reversion jump-diffusion models for modeling energy commodity spot prices with jumps and spikes. We demonstrate how the prices of the energy commodity derivatives can be obtained by means of transform analysis. The market anticipation of jumps and spikes in the commodity spot price process explains the enormous implied volatility observed from market prices of traded energy commodity options. By using the proper jump-diffusion models, as opposed to the geometric Brownian motion (GBM) model, for modeling energy commodity spot prices, we obtain prices of short-maturity out-of–the-money options that are closer to market price data. We showed that energy commodity options can be used to value the capacity of real assets. The implication of jumps and spikes to capacity valuation in the context of electric power industry is that, in the near-term when the effects of jumps and spikes are significant, even a very inefficient power plant is quite valuable. This might explain why recently sold power plants fetched hefty premia over book value. While the upwards jumps and spikes increase the options values embedded in the installed capacity, we illustrated that the presence of downwards jumps in the value process of an investment reduces the value of the opportunity to invest and induces firms to wait shorter before they invest. We also mentioned briefly how we can fit the models using electricity price data but a formal parameter estimation was out of the scope of this work. As for future research, we feel it is important to develop an efficient econometric model to perform a rigorous parameter estimation as more electricity market data becomes available.

A Proof of Proposition

Proof. The transform function φ_{1a} in **Model 1a** is given by (3.5) and (3.6). For the case of S_T^1, we have $\varphi(0, \overline{X}_t, t, T) = e^{-r(T-t)}$ and

$$\begin{aligned} e^{r(T-t)}\varphi(i \cdot w\overline{e}_1, \overline{X}_t, t, T)e^{-iw\ln v} &= \exp[i \cdot wX_t \exp(-\kappa_1 \tau) + \frac{a_1 \sigma_1^2(-w^2)}{4\kappa_1} \\ &\quad + i \cdot w\theta_1(1 - \exp(-\kappa_1 \tau)) - j(i \cdot w, \tau) - i \cdot w \ln v] \end{aligned}$$

where $a_1 = 1 - \exp(-2\kappa_1 \tau)$ and $j(i \cdot w, \tau) = \sum_{j=1}^{2} \frac{\lambda_J^j}{\kappa_1} \ln \frac{i \cdot w\mu_J^j - 1}{i \cdot w\mu_J^j \exp(-\kappa_1 \tau) - 1}$.

Next, we spell out $Im[e^{r(T-t)}\varphi(i \cdot w\overline{e}_1, \overline{X}_t, t, T)e^{-iw\ln v}]$ as

$$Im[e^{r\tau}\varphi(i \cdot w\overline{e}_1, \overline{X}_t, t, T)e^{-iw\ln v}] = g(w; \sigma_1, \lambda_J^j, \mu_J^j) \sin(m(X_t, \theta_1) \cdot w + n(w; \lambda_J^j, \mu_J^j))$$

where

$$
\begin{aligned}
g(w;\sigma_1,\lambda_J^j,\mu_J^j) &= \exp[\frac{a_1\sigma_1^2(-w^2)}{4\kappa_1} + \sum_{j=1}^{2}\frac{\lambda_J^j}{\kappa_1}\ln\frac{1+w^2(\mu_J^j)^2e^{-2\kappa_1\tau}}{\sqrt{(1+w^2(\mu_J^j)^2e^{-2\kappa_1\tau})^2+w^2(\mu_J^j)^2(1-e^{-\kappa_1\tau})^2}}] \\
m(X_t,\theta_1) &= X_t\exp(-\kappa_1\tau)+\theta_1(1-\exp(-\kappa_1\tau))-\ln v \\
n(w;\lambda_J^j,\mu_J^j) &= \sum_{j=1}^{2}\frac{\lambda_J^j}{\kappa_1}\arctan(\frac{w\mu_J^j(1-e^{-\kappa_1\tau})}{1+w^2(\mu_J^j)^2e^{-\kappa_1\tau}})
\end{aligned}
$$

To establish S_T^1 is stochastically increasing in X_t under technical conditions for λ_J^j and μ_J^j, we denote $Im[e^{r\tau}\varphi(i\cdot w\overline{e}_1,\overline{X}_t,t,T)e^{-iw\ln v}]$ by $f(w)$ and examine $\frac{df}{dX_t}$.

$$
\frac{df}{dX_t} = g(w;\sigma_1,\lambda_J^j,\mu_J^j)\cos(m(X_t,\theta_1)\cdot w + n(w;\lambda_J^j,\mu_J^j))\cdot\exp(-\kappa_1\tau)w
$$

Note that $g(\cdot)$ is a positive and decreasing function in w. Under technical conditions for λ_J^j and μ_J^j such that

$$
\int_0^\infty g(w)\cos(m(X_t,\theta_1)\cdot w + n(w;\lambda_J^j,\mu_J^j))dw \geq 0
$$

for example (A) is true when $n(w;\lambda_J^j,\mu_J^j)\equiv 0$, we have

$$
\begin{aligned}
\frac{d}{dX_t}[\int_0^\infty \frac{Im[\varphi(i\cdot w\overline{e}_i,\overline{X}_t,t,T)e^{r\cdot\tau-i\cdot w\ln v}]}{w}dw] &= \int_0^\infty g(w)\cos(m(X_t,\theta_1)\cdot w + n(w;\lambda_J^j,\mu_J^j))dw \\
&\geq 0
\end{aligned}
$$

Therefore $CF_{S_T^1}(v)$ is decreasing in X_t for any v, i.e. S_T^1 is stochastically increasing in X_t. The rest of the proposition can be similarly proved. ∎

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